

# A Graph Theoretic Method for Determining Generating Sets of Prime Ideals in $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{C}))$

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## Abstract

We take a graph theoretic approach to the problem of finding generators for those prime ideals of  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{C}))$  which are invariant under the torus action  $(\mathbb{C}^*)^{m+n-1}$ . Launois [12] has shown that the generators consist of certain quantum minors of the matrix of canonical generators of  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{C}))$  and in [11] gives an algorithm to find them. In this paper we modify a classic result of Lindström [14] and Gessel-Viennot [6] to show that a quantum minor is in the generating set for a particular ideal if and only if we can find a particular set of vertex-disjoint directed paths in an associated directed graph.

## 1 Introduction

Recent attention has focused on understanding the structure of the set of prime ideals of the quantized coordinate ring of  $m \times n$  matrices over  $\mathbb{C}$ . We

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denote this algebra by  $\mathcal{A} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{C}))$ . Goodearl and Letzter [9] have developed a powerful stratification theory that allows one to restrict attention to those prime ideals that are held stable under the action of an algebraic torus  $\mathcal{H} = (\mathbb{C}^*)^{m+n-1}$ . Call the set of such ideals  $\mathcal{H}\text{-spec}(\mathcal{A})$ .

Cauchon [4] applied his deleting derivations algorithm to  $\mathcal{A}$  and obtained a bijection between  $\mathcal{H}\text{-spec}(\mathcal{A})$  and a set of combinatorial objects, which are called *Cauchon diagrams*. A Cauchon diagram consists of an  $m \times n$  grid of squares, each square coloured black or white so that for any black square, either every square above it or every square to its left is also black. An example appears in Figure 1.

Launois [12] further developed these ideas and was able to prove a conjecture of Goodearl and Lenagan [8] that the ideals of  $\mathcal{H}\text{-spec}(\mathcal{A})$  are generated by quantum minors of the canonical matrix of generators of  $\mathcal{A}$ . Furthermore, Launois [11] gave an algorithm which explicitly determines the quantum minors in question. The first step of this algorithm is to calculate a certain matrix  $T$  with entries in a McConnell-Pettit algebra. The next step is to determine the vanishing minors of this matrix.

The main contribution of this paper is Theorem 5.6 which essentially states that Launois's algorithm is equivalent to finding certain sets of disjoint paths in a Cauchon diagram. Roughly speaking, we show that a  $k \times k$  submatrix of  $T$  has non-vanishing quantum determinant if and only if we can find a corresponding set of  $k$  disjoint paths in the Cauchon diagram. For example, Figure 1 gives an example of a  $4 \times 5$  Cauchon diagram with two paths drawn over top; the existence of these paths implies that the minor of  $T$  indexed by  $\{1, 2\} \times \{1, 2\}$  is non-zero. This method was inspired by an old result of Lindström [14] and Gessel-Viennot [6]. We will call this result Lindström's Lemma (see [1] for an excellent exposition). A major component of our work (Theorem 4.4) is the proof of a  $q$ -analogue of a special case of their classic lemma.

Cauchon diagrams essentially appeared independently in the work of Postnikov [15] on totally nonnegative Grassmannians where they are known as *J-diagrams* (also sometimes as *Le-diagrams*). His work implies a correspondence between Cauchon diagrams and the collection of totally non-negative matrices over  $\mathbb{R}$  (that is, matrices all of whose minors are non-negative). The connection between Postnikov's work and the ideal structure of  $\mathcal{A}$  has been developed by Goodearl, Launois and Lenagan [7]. In particular, they have independently derived a new approach to finding a generating set for a member of  $\mathcal{H}\text{-spec}(\mathcal{A})$  corresponding to a given Cauchon diagram.

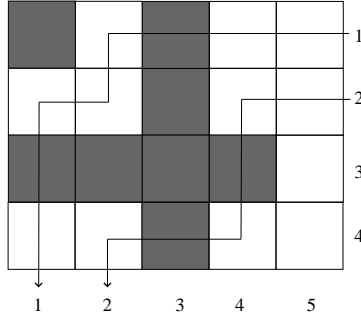


Figure 1: A Cauchon diagram together with two disjoint paths from rows 1 and 2 to columns 1 and 2.

In view of this and the results of this paper, it is perhaps not surprising that Talaska [16] has independently been able to give an explicit description of Postnikov's J-diagram totally-nonnegative-matrix correspondence using the classic version of Lindström's Lemma.

## 2 Background

### 2.1 Basic Definitions

In this section we give the definitions which are of relevance to us and outline some of the basic results in this area.

**Conventions and Notation 2.1.** Below is a list of notation and conventions which will be used in this work.

- If  $n$  is a positive integer, then  $[n] := \{1, 2, \dots, n\}$ .
- As we will be working in a noncommutative algebra, we here clarify that the standard product notation of a set of elements, say  $\prod_{i=1}^n z_i$  is the ordered product, from *left to right* of the  $z_i$ , i.e.,

$$\prod_{i=1}^n z_i = z_1 z_2 \cdots z_n.$$

- If  $\sigma$  is a permutation acting on a finite set of integers  $I$  then the *length*  $l(\sigma)$  denotes the total number of pairs  $i, j \in I$  such that  $i < j$  and  $\sigma(i) > \sigma(j)$ . Such a pair is called an inversion.

- If  $M$  is an  $m \times n$  matrix and  $I \subseteq [m]$  and  $J \subseteq [n]$ , denote by  $M[I, J]$  the submatrix of  $M$  whose rows are indexed by  $I$  and columns indexed by  $J$ .
- The *quantum determinant*, or *q-determinant* of a  $k \times k$  matrix  $M$  with respect to  $q \in \mathbb{C}$  is the quantity

$$\det_q(M) := \sum_{\sigma \in S_k} (-q)^{l(\sigma)} \prod_{i=1}^k M[i, \sigma(i)].$$

- We will use the standard partial ordering on pairs of integers by setting  $(i, j) \leq (s, t)$  if and only if  $i \leq s$  and  $j \leq t$ .
- We also totally order the set  $([m] \times [n]) \cup (m, n+1)$  by setting  $(i, j) \preceq (s, t)$  if and only if either  $i < s$  or  $i = s$  and  $j \leq t$ . This is called the *lexicographic* ordering.
- Let  $(i, j) \in ([m] \times [n]) \cup (m, n+1)$ . If  $(i, j) \neq (m, n+1)$ , we denote by  $(i, j)^-$  the greatest element less than  $(i, j)$  in the lexicographic ordering. If  $(i, j) \neq (1, 1)$ , we denote by  $(i, j)^+$  the least element which is greater than  $(i, j)$  in this ordering.

We assume the reader is familiar with the elementary definitions of graphs and directed graphs but we refer the reader to, for example, [17] for further details.

The algebraic structure of interest to us is defined as follows.

**Definition 2.2.** Fix  $q \in \mathbb{C}^*$  and two positive integers  $m$  and  $n$ . The ring  $\mathcal{A} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{C}))$  is the quantized coordinate ring of  $m \times n$  matrices over  $\mathbb{C}$ . In other words,  $\mathcal{A}$  is the  $\mathbb{C}$ -algebra generated by the  $mn$  indeterminants  $x_{i,j}$  (the canonical generators) which satisfy the following relations. Consider the  $m \times n$  matrix  $X_{\mathcal{A}}$  where  $X_{\mathcal{A}}[i, j] = x_{i,j}$ . Then for any  $2 \times 2$  submatrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , the following hold:

1.  $ab = qba$  and  $cd = qdc$ ,
2.  $ac = qca$  and  $bd = qdb$ ,
3.  $bc = cb$ ,

$$4. \ ad - da = (q - q^{-1})bc.$$

**Remark 2.3.** In this work we always fix  $q \in \mathbb{C}^*$  to be transcendental over  $\mathbb{Q}$

It is known that  $\mathcal{A}$  is Noetherian and since  $q$  is not a root of unity, every prime ideal is completely prime (see [8]). Furthermore,  $\mathcal{A}$  is a domain of finite GK dimension. We may therefore conclude by Proposition 4.13 in [10] the existence of the quotient division algebra  $Q(\mathcal{A})$ . Denote by  $\text{spec}(\mathcal{A})$  the set of prime ideals in  $\mathcal{A}$ .

Although our results are to be applied to  $\mathcal{A}$ , we will in fact be working in the following related algebra  $\mathcal{B}$ , which is a McConnell-Pettit algebra.

**Definition 2.4.** We denote by  $\mathcal{B}$  the  $\mathbb{C}$ -algebra generated by the division ring  $\langle t_{i,j} | i \in [m], j \in [n] \rangle$  where commutation amongst the canonical generators  $t_{i,j}$  is defined as follows. Consider the  $m \times n$  matrix  $T_{\mathcal{B}}$  where  $T_{\mathcal{B}}[i, j] = t_{i,j}$ . Then for any  $2 \times 2$  submatrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , the following hold:

1.  $ab = qba$  and  $cd = qdc$ ,
2.  $ac = qca$  and  $bd = qdb$ ,
3.  $bc = cb$ ,
4.  $ad = da$ .

Consider the following set of automorphisms of  $\mathcal{A}$  (if  $m \neq n$  these are the only ones [13]). Let  $h = (\rho_1, \dots, \rho_m, \gamma_1, \dots, \gamma_n) \in (\mathbb{C}^*)^{m+n}$ . Then for any canonical generator set  $x_{i,j}$ ,

$$h \cdot x_{i,j} := \rho_i \gamma_j x_{i,j}.$$

Notice that for any nonzero  $\rho$ , the action  $(\rho, \rho, \dots, \rho, \rho^{-1}, \dots, \rho^{-1})$  acts trivially on  $\mathcal{A}$ . So we need only consider the algebraic torus action  $\mathcal{H} \simeq \mathbb{C}^{m+n-1}$  of  $\mathcal{A}$ .

An ideal  $I$  is  $\mathcal{H}$ -invariant if  $h \cdot I = I$  for all  $h \in \mathcal{H}$ . The (finite) set of all  $\mathcal{H}$ -invariant prime ideals is denoted by  $\mathcal{H}\text{-spec}(\mathcal{A})$ . The results of Goodearl and Letzter [9] imply the following.

**Theorem 2.5** ( $\mathcal{H}$ -stratification of  $\text{spec}(\mathcal{A})$ ). *The set  $\text{spec}(\mathcal{A})$  can be partitioned (or stratified) into a finite, disjoint union as follows.*

$$\text{spec}(\mathcal{A}) = \bigcup_{J \in \mathcal{H}\text{-spec}(\mathcal{A})} Y_J,$$

where  $Y_J := \{P \in \text{spec}(\mathcal{A}) \mid \cap_{h \in \mathcal{H}} h \cdot P = J\}$ .

## 2.2 Cauchon Diagrams

We begin this section by briefly describing Cauchon's deleting derivations algorithm [5] applied to  $\mathcal{A}$  (recall Definition 2.2).

First define  $X^{(m,n+1)} := X_{\mathcal{A}}$ , where  $X^{(m,n+1)}$  is a matrix with entries in the quotient division algebra  $Q(\mathcal{A})$ . Now assuming that  $X^{(s,t)+}$  has been constructed we define  $X^{(s,t)}$  as follows.

If  $(i, j) \not\leq (s, t)$  then  $X^{(s,t)}[i, j] = X^{(s,t)+}[i, j]$ .

For each  $(i, j) < (s, t)$ , we consider the  $2 \times 2$  submatrix

$$X^{(s,t)+}[\{i, s\}, \{j, t\}] = \begin{bmatrix} X^{(s,t)+}[i, j] & X^{(s,t)+}[i, t] \\ X^{(s,t)+}[s, j] & X^{(s,t)+}[s, t] \end{bmatrix} := \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

Then we set

$$X^{(s,t)}[i, j] = x - yw^{-1}z \in Q(\mathcal{A}).$$

Let  $\mathcal{A}^{(s,t)} \subseteq Q(\mathcal{A})$  denote the algebra whose matrix of canonical generators is  $X^{(s,t)}$ . Cauchon [4] has shown that there exists an embedding  $\text{spec}(\mathcal{A}^{(s,t)+}) \hookrightarrow \text{spec}(\mathcal{A}^{(s,t)})$ . Furthermore, the final algebra  $\mathcal{A}^{(1,1)}$  is isomorphic to the McConnell-Pettit algebra  $\mathcal{B}$ . The composition of all such embeddings gives an embedding  $\psi : \text{spec}(\mathcal{A}) \hookrightarrow \text{spec}(\mathcal{B})$ .

Finally, he showed that the image of  $\mathcal{H}\text{-spec}(\mathcal{A})$  under  $\psi$  is parametrized by a useful set of combinatorial objects, which we call Cauchon diagrams.

**Definition 2.6.** Let  $\mathcal{C}$  be an  $m \times n$  grid of squares where we have coloured each square either white or black. Call  $\mathcal{C}$  a *Cauchon diagram* if, for every black square, either every square above it or every square to its left is also black. Take  $W_{\mathcal{C}}$  to be the set of white squares and  $B_{\mathcal{C}}$  to be the set of black squares of a Cauchon diagram  $\mathcal{C}$ .

Index the squares of an  $m \times n$  Cauchon diagram as one would a matrix. That is, the square in the  $i$ th row from the top and  $j$ th column from the left is the  $(i, j)$  square. We have already seen an example of a Cauchon diagram in Figure 1.

### 3 Cauchon Graphs

We essentially follow Postnikov [15] by defining a weighted directed graph given the Cauchon diagram  $\mathcal{C}$ . If  $(i, j)$  is a white square in  $\mathcal{C}$ , let be  $(i, j^-)$  the first white square to its left (if one exists) and  $(i, j^+)$  the first white square to its right (if one exists). Similarly, let  $(i^+, j)$  be the first white square below  $(i, j)$  (if one exists).

**Definition 3.1.** Let  $\mathcal{C}$  be an  $m \times n$  Cauchon diagram. The *Cauchon graph*  $\mathcal{G}_{\mathcal{C}} = (V, \vec{E}, w)$  is an edge-weighted directed graph defined as follows. The vertices consist of the set  $V = W_{\mathcal{C}} \cup \{r_1, \dots, r_m\} \cup \{c_1, \dots, c_n\} := W_{\mathcal{C}} \cup R \cup C$ . The set  $\vec{E}$  of directed edges, together with a weight function  $w : \vec{E}(\mathcal{G}_{\mathcal{C}}) \rightarrow \mathcal{B}$ , are constructed as follows.

1. For every  $i \in [m]$  we put a directed edge from  $r_i$  to the rightmost white square in row  $i$  (if it exists), say  $(i, k)$ . Give these edges weight  $t_{i,k}$ .
2. For every column  $j \in [n]$  we make a directed edge from the bottom-most white square in column  $j$  (if it exists) to the vertex  $c_j$ . Give these edges weight 1.
3. For every  $(i, j) \in W_{\mathcal{C}}$  we make a directed edge from  $(i, j)$  to  $(i, j^-)$  (if it exists). Give these edges a weight  $t_{i,j}^{-1}t_{i,j^-}$ .
4. For every  $(i, j) \in W_{\mathcal{C}}$  we make a directed edge from  $(i, j)$  to  $(i^+, j)$  (if it exists). Give each of these edges a weight of 1.

For convenience, we always assume a Cauchon graph is embedded in the plane in the following way. First place a vertex in each white square of  $\mathcal{C}$  (labelled by the entry of the white square). Next place a vertex to the right of each row and below each column. The vertex to the right of row  $i$  is labelled  $r_i$ , and the vertex below column  $j$  is  $c_j$ . See Figure 2. Thus we will unambiguously use directional terms such as horizontal, vertical, above, below, left and right when discussing a Cauchon graph.

**Notation 3.2.** Here we list some useful notation and conventions for a Cauchon graph corresponding to an  $m \times n$  Cauchon diagram  $\mathcal{C}$ .

- If  $I \subseteq [m]$  and  $J \subseteq [n]$ , then  $R_I = \{r_i \in R \mid i \in I\}$  and  $C_J = \{c_j \in C \mid j \in J\}$ .

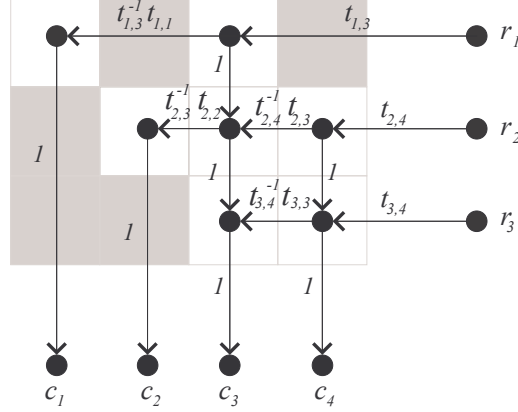


Figure 2: A Cauchon graph superimposed on top of its Cauchon diagram.

- Let  $e = ((i, k), (i, j))$  be a horizontal edge with both endpoints in  $W_C$  (so  $k > j$ ). Then we let  $\text{row}(e) = i$ ,  $\text{col}_1(e) = k$  and  $\text{col}_2(e) = j$ . The pair  $\{k, j\}$  will be denoted by  $\text{col}(e)$ . In other words,  $\text{col}_1(e)$  is the column containing the right end (or tail) of  $e$  and  $\text{col}_2(e)$  is the column containing the left end (or head) of  $e$ .
- All paths in this paper are assumed to be *directed* paths. If  $P = (v_0, e_1, v_1, e_2, \dots, e_n, v_n)$  is a directed path in a Cauchon graph then we may omit the edges and write  $P$  as  $v \rightarrow v_1 \rightarrow \dots \rightarrow v_n$ . When we wish to emphasize the endpoints of  $P$  we will write  $P : v_0 \Rightarrow v_n$ .
- The weight of a path  $P = (v_0, e_1, v_1, e_2, \dots, e_n, v_n)$  is the product of the weights of its edges, multiplied from left to right in the order they appear in  $P$ . In other words,

$$w(P) = w(e_1)w(e_2) \cdots w(e_n) = \prod_{i=1}^n w(e_i).$$

- If  $K : v_0 \Rightarrow v$  and  $L : v \Rightarrow v_n$  are two paths in a Cauchon graph, then we write  $KL = KL : v_0 \Rightarrow v_n$  for the path obtained by appending  $L$  to  $K$  in the obvious way. That is,  $KL$  is the path which travels along  $K$  from  $v_0$  to  $v$ , and then continues on  $L$  from  $v$  to  $v_n$ . Note that  $KL$  is still a directed path since Cauchon graphs are acyclic by the next proposition.



**Proposition 3.3.** *Let  $\mathcal{C}$  be a Cauchon diagram. Then the graph  $\mathcal{G}_{\mathcal{C}}$  has the following properties:*

1. *It is acyclic (i.e. it has no directed cycles) and,*
2. *The embedding described above is a planar embedding (i.e. no edges touch except at a vertex).*
3. *The completely horizontal path  $P : (i, j_1) \Rightarrow (i, j_2)$  has weight exactly  $t_{i, j_2}^{-1} t_{i, j_1}$*

*Proof.* Since all edges are directed either right to left or top to bottom the first property is easy to see. To see planarity, suppose two edges cross. Then these edges must consist of one vertical and one horizontal edge and their intersection point corresponds to a black square. But then this implies that we have a black square in  $\mathcal{C}$  with a white square above and a white square to the left, contradicting the definition of a Cauchon diagram.

Finally if  $(i, k)$  is an internal vertex of  $P : (i, j_1) \Rightarrow (i, j_2)$  then  $(i, k^-)$  and  $(i, k^+)$  exist. But the edge  $((i, k^+), (i, k)) \in P$  has weight  $e_1 = t_{i, k^+}^{-1} t_{i, k}$  and the edge  $e_2 = ((i, k), (i, k^-)) \in P$  has weight  $t_{i, k}^{-1} t_{i, k^-}$ . Now  $w(e_1)w(e_2)$  appears in  $w(P)$  and therefore equals  $t_{i, k^+}^{-1} t_{i, k^-}$ .

In other words an internal vertex  $(i, k)$  never appears as a subscript in a term of  $w(P)$  since  $t_{i, k}$  always ends up next to  $t_{i, k}^{-1}$ . It follows that  $w(P) = t_{i, j_2}^{-1} t_{i, j_1}$ .  $\square$

Since the edge weights are in a non-commutative ring, the remainder of this section is devoted to a sequence of lemmas which give commutation relations between edges and between certain paths in a Cauchon Graph.

**Lemma 3.4.** *Let  $\mathcal{C}$  be a Cauchon diagram. Let  $e$  and  $f$  be horizontal edges in  $\mathcal{G}_{\mathcal{C}}$  with both endpoints in  $W$  and such that  $\text{row}(f) \leq \text{row}(e)$ .*

1. *If  $\text{col}(e) \cap \text{col}(f) = \emptyset$ , then  $w(f)w(e) = w(e)w(f)$ .*
2. *If  $|\text{col}(e) \cap \text{col}(f)| = 1$ , then:*
  - i.  *$w(f)w(e) = q w(e)w(f)$ , if  $\text{col}_i(e) = \text{col}_i(f)$  for  $i = 1$  or  $i = 2$ , and*
  - ii.  *$w(f)w(e) = q^{-1} w(e)w(f)$  otherwise.*
3. *If  $|\text{col}(e) \cap \text{col}(f)| = 2$ , then  $w(f)w(e) = q^2 w(e)w(f)$ .*

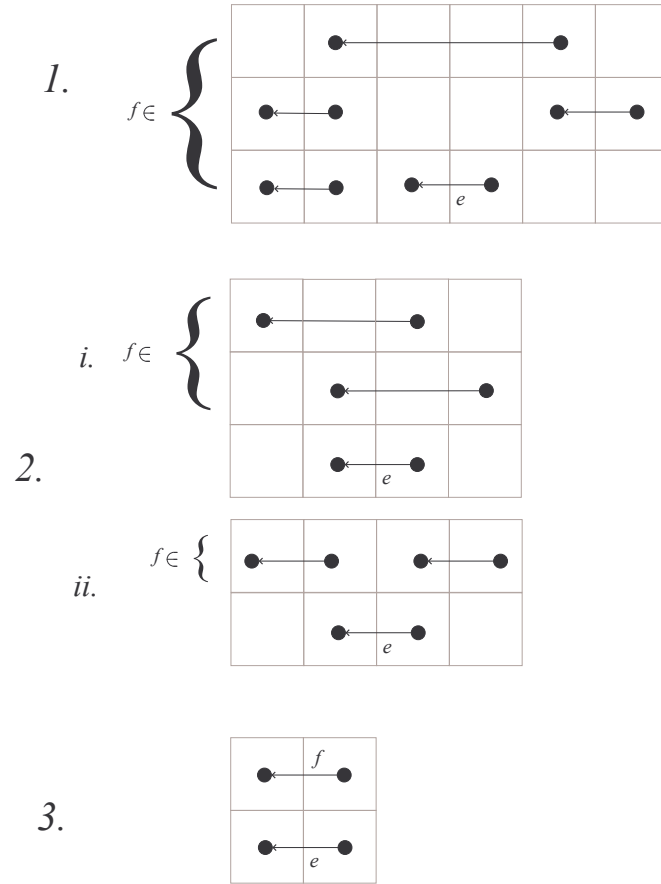


Figure 3: Examples of the different cases in Lemma 3.4.

*Proof.* We verify the case  $\text{col}(e) \cap \text{col}(f) = \emptyset$ . The other cases can be disposed of in a similar manner by checking the possibilities (see Figure 3). So suppose  $\text{col}(e) \cap \text{col}(f) = \emptyset$ .

First note that if  $j$  is such that  $\text{col}_2(e) < j < \text{col}_1(e)$  then it follows that entry  $(\text{row}(e), j)$  is black in  $\mathcal{C}$ , and since  $(\text{row}(e), \text{col}_2(e))$  is a white square to its left we must have that  $(i, j)$  is black for every  $i \leq \text{row}(e)$ . In other words, no horizontal edge in  $K$  has an endpoint whose column coordinate lies strictly between the column coordinates of  $e$ .

Now if  $\text{row}(e) \neq \text{row}(f)$  then  $w(e)$  and  $w(f)$  clearly commute by the definition of the algebra  $\mathcal{B}$ . So suppose  $\text{row}(e) = \text{row}(f)$ . Say  $w(e) = t_{i,j_1}^{-1} t_{i,j_2}$  and  $w(f) = t_{i,j_3}^{-1} t_{i,j_4}$  where  $j_1 > j_2 > j_3 > j_4$ . Then

$$\begin{aligned} w(e)w(f) &= (t_{i,j_1}^{-1} t_{i,j_2})(t_{i,j_3}^{-1} t_{i,j_4}) \\ &= (q^{-1}q) t_{i,j_3}^{-1} (t_{i,j_1}^{-1} t_{i,j_2}) t_{i,j_4} \\ &= (q^{-1}q)(qq^{-1}) (t_{i,j_3}^{-1} t_{i,j_4})(t_{i,j_1}^{-1} t_{i,j_2}) \\ &= w(f)w(e). \end{aligned}$$

□

We remark that Lemma 3.4, parts (1) and (2) remain true if  $e$  or  $f$  is an edge which has an endpoint in  $R$ .

**Lemma 3.5.** *Let  $K : v_0 \Rightarrow v$  and  $L : v \Rightarrow v_t$  be directed paths in a Cauchon graph.*

1. *If either  $K$  or  $L$  contain only vertical edges then  $w(K)w(L) = w(L)w(K)$ .*
2. *If both  $K$  and  $L$  contain a horizontal edge, then  $w(K)w(L) = q^{-1} w(L)w(K)$ .*

*Proof.* We need only consider the horizontal edges of  $K$  and  $L$  since all vertical edges have weight 1 and so commute with everything. So if either  $K$  or  $L$  contain only vertical edges then  $w(K) = 1$  or  $w(L) = 1$  and  $w(K)$  and  $w(L)$  commute. So suppose both  $K$  and  $L$  contain horizontal edges. Let  $k$  be the last horizontal edge in  $K$  and let  $l$  be the first horizontal edge in  $L$ .

Now by the embedding of a Cauchon graph, the horizontal edges of  $L$  are always to the left of horizontal edges of  $K$  or “south-west” of  $K$ . So by Lemma 3.4 (1), in computing  $w(K)w(L)$  the only edge weights which do not commute are those of  $k$  and  $l$ .

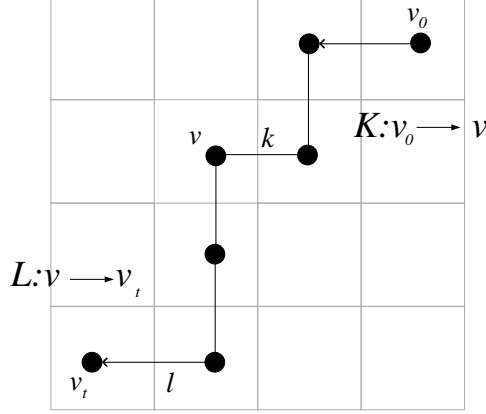


Figure 4: Example of a situation in Lemma 3.5.

$$\begin{aligned}
w(K)w(L) &= w(K \setminus \{k\})w(k)w(l)w(L \setminus \{l\}) \\
&= q^{-1} w(K \setminus \{k\})w(l)w(k)w(L \setminus \{l\}) \\
&= q^{-1} w(l)w(L \setminus \{l\})w(K \setminus \{k\})w(k) \\
&= q^{-1} w(L)w(K)
\end{aligned}$$

□

**Lemma 3.6.** *Let  $\mathcal{G}_C$  be a Cauchon graph and let  $K : v \Rightarrow c_i$  and  $L : v \Rightarrow c_j$  be two directed paths with only their initial vertex in common. Let  $K$  be the path that starts with a horizontal edge and  $L$  be the path that starts with a vertical edge. Then the weights of the two paths commute as follows.*

1. *If  $L$  consists only of vertical edges then  $w(K)w(L) = w(L)w(K)$ .*
2. *If  $L$  has a horizontal edge then  $w(K)w(L) = q w(L)w(K)$ .*

*Proof.* Case 1 is obvious since here we have  $w(L) = 1$ . So we may suppose  $L$  has at least one horizontal edge. By Lemma 3.4, any horizontal edge  $e$  in  $L$  commutes with any edge in  $K$  except those whose column coordinates intersect the set  $\text{col}(e)$ . Recall from the proof of Lemma 3.4 that no edge in  $K$  has an endpoint in between (with respect to column coordinates) the endpoints of edges in  $L$ .

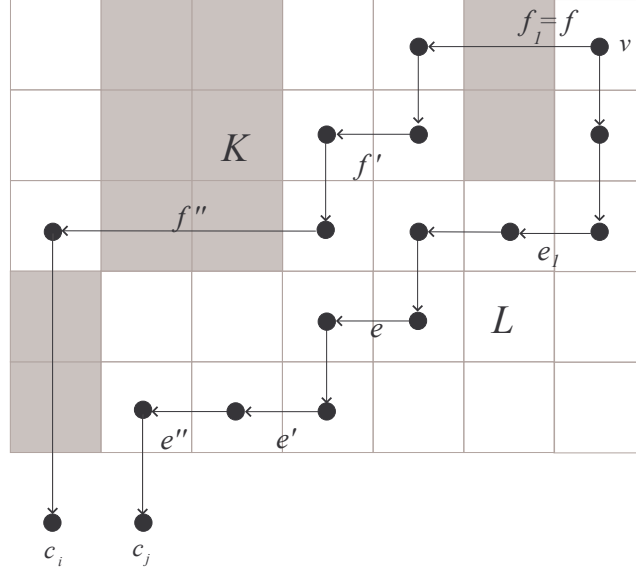


Figure 5: Example of a typical situation of Lemma 3.6.

Suppose  $f_1$  is the first horizontal edge in  $K$  and  $e_1$  is the first horizontal edge in  $L$ . Then we have  $\text{col}_2(e_1) = \text{col}_2(f_1)$ . There are two cases to consider (see Figure 5).

**Case (i):**  $\text{col}_1(f_1) < \text{col}_1(e_1)$ . Then by Lemma 3.4 (2i),  $w(f_1)w(e_1) = qw(e_1)w(f_1)$ .

**Case (ii):**  $\text{col}_1(f_1) = \text{col}_1(e_1)$ . Then the second horizontal edge  $f_2$  of  $K$  satisfies  $\text{col}_2(f_2) = \text{col}_1(e_1)$  and  $\text{col}_1(f_2) < \text{col}_1(e_1)$ . Applying Lemma 3.4 twice we find

$$\begin{aligned} w(f_1)w(f_2)w(e_1) &= w(f_1)q^{-1}w(e_1)w(f_2), \text{ by Lemma 3.4 (2.i),} \\ &= (q^{-1}q^2)w(e_1)w(f_1)w(f_2), \text{ by Lemma 3.4 (3),} \\ &= qw(e_1)w(f_1)w(f_2). \end{aligned}$$

It follows that  $w(K)w(e_1) = qw(e_1)w(K)$ .

Now suppose  $e$  is *not* the first horizontal edge in  $L$ . Then by Lemma 3.4 (1),  $e$  commutes with those edges  $f \in K$  such that  $\text{col}(f) \cap \text{col}(e) = \emptyset$ . On the other hand, for any such edge, exactly one of the following three cases hold.

**Case (a):** The edge  $f''$  which lies directly above  $e$  satisfies  $\text{col}_2(f'') < \text{col}_2(e)$  and  $\text{col}_1(f'') > \text{col}_1(e)$ . In other words, *every* edge of  $K$  is covered by Lemma 3.4 (1). For an example of this case, see edge  $e''$  in Figure 5. It follows that  $e$  commutes with  $w(K)$ .

**Case (b):** There exist two distinct edges  $f'$  and  $f''$  such that  $|\text{col}(f') \cap \text{col}(e)| = 1$  and  $|\text{col}(f'') \cap \text{col}(e)| = 1$ . For an example of this case, see edge  $e'$  in Figure 5. So we have that  $f'$  and  $f''$  are both covered by Lemma 3.4 (2). Furthermore, exactly one is in subcase (2i) of that lemma, and the other is in subcase (2ii). Since both possibilities are handled similarly we only show the situation where  $f'$  is in case (2ii) of Lemma 3.4. We have

$$\begin{aligned} w(f'')w(f')w(e) &= w(f'')q^{-1}w(e)w(f') \\ &= qw(e)w(f'')q^{-1}w(e)w(f') \\ &= w(e)w(f'')w(f'). \end{aligned}$$

**Case (c):** There exist three distinct edges  $f, f'$  and  $f''$  in  $K$  such that  $f'$  is covered by Lemma 3.4 (3), while both  $f$  and  $f''$  are covered by Lemma 3.4 (2ii). For an example of this case, see Figure 5. Then

$$\begin{aligned} w(f'')w(f')w(f)w(e) &= w(f'')w(f')q^{-1}w(e)w(f) \\ &= w(f'')q^2w(e)w(f')q^{-1}w(f) \\ &= q^{-1}w(e)w(f'')q^2w(f')q^{-1}w(f) \\ &= w(e)w(f'')w(f')w(f). \end{aligned}$$

Putting Cases (a), (b) and (c) together, we get that if  $e$  is not the first horizontal edge of  $L$ , then  $w(K)w(e) = w(e)w(K)$ . Hence

$$\begin{aligned} w(K)w(L) &= w(K)w(e_1)w(L \setminus e_1) \\ &= qw(e_1)w(K)w(L \setminus e_1) \\ &= qw(e_1)w(L \setminus e_1)w(K) \\ &= qw(L)w(K). \end{aligned}$$

□

## 4 Vertex Disjoint Path Systems and $q$ -Determinants

In this section we prove one of our main tools for proving Theorem 5.6. We begin with some definitions.

**Definition 4.1.** Let  $\mathcal{C}$  be an  $m \times n$  Cauchon diagram. Let  $I = \{i_1, \dots, i_k\} \subseteq [m]$  and  $J = \{j_1, \dots, j_k\} \subseteq [n]$  be two subsets of equal size.

An  $(R_I, C_J)$ -path system  $\mathcal{P}$  is a set of  $k$  directed paths in  $\mathcal{G}_{\mathcal{C}}$ , each starting at different a vertex in  $R_I$  and each ending at a different vertex in  $C_J$ . Furthermore:

- There exists a permutation  $\sigma_{\mathcal{P}} \in S_k$  such that

$$\mathcal{P} = \{P_l : r_{i_l} \Rightarrow c_{j_{\sigma_{\mathcal{P}}(l)}} \mid l \in [k]\}$$

- The  $q$ -sign of  $\mathcal{P}$  is the quantity

$$\text{sgn}_q(\mathcal{P}) = (-q)^{l(\sigma_{\mathcal{P}})},$$

where we recall that  $l(\sigma_{\mathcal{P}})$  is the length of the permutation  $\sigma_{\mathcal{P}}$  as defined in Conventions and Notation 2.1.

- A path system is *vertex disjoint* if no two paths share a vertex.
- The weight of  $\mathcal{P}$  is the product  $\prod_{l=1}^k w(P_l) = w(P_1)w(P_2) \cdots w(P_k)$ .

We will need the following easy lemma.

**Lemma 4.2.** Let  $\mathcal{C}$  be a Cauchon diagram. Let  $I \subseteq [m]$  and  $J \subseteq [n]$  be two sets of cardinality  $k$ , and suppose  $\mathcal{P} = \{P_1, \dots, P_k\}$  is a non-vertex disjoint  $(R_I, C_J)$ -path system in  $\mathcal{G}_{\mathcal{C}}$ . Then there exists an  $i$  such that  $P_i$  and  $P_{i+1}$  share a vertex.

*Proof.* Let

$$d = \min\{|i - j| \mid i \neq j \text{ and } P_i \text{ and } P_j \text{ share a vertex}\}.$$

Observe  $d$  is well defined and at least 1 since there exists at least one pair of intersecting paths in  $\mathcal{P}$ . Let  $P_i$  and  $P_j$  be two paths which achieve this minimum. Hence there is a first vertex  $x \in W_{\mathcal{C}}$  at which they intersect. If  $r_i$  and  $r_j$  are the first vertices on the paths  $P_i$  and  $P_j$  respectively, then the





where the sum is over all **vertex disjoint**  $(R_I, C_J)$ -path systems in  $\mathcal{G}_C$ .

*Proof.* In order to simplify the presentation of this proof we will take  $I = J = \{1, \dots, k\}$ . The proof of the general case is essentially the same but notationally more cumbersome.

First, note that we have

$$\begin{aligned} \det_q(M_C[I, J]) &= \sum_{\sigma \in S_k} \text{sgn}_q(\sigma) \left( \prod_{i=1}^k M_C[i, \sigma(i)] \right) \\ &= \sum_{\sigma} \text{sgn}_q(\sigma) \left( \prod_{i=1}^k \left( \sum_{P: r_i \Rightarrow c_{\sigma(i)}} w(P) \right) \right) \\ &= \sum_{(R_I, C_J)\text{-path systems } \mathcal{P}} \text{sgn}_q(\mathcal{P}) w(\mathcal{P}). \end{aligned}$$

Let  $\mathcal{N}$  be the set of *non*-vertex disjoint  $(R_I, C_J)$ -path systems. We first prove that

$$\sum_{\mathcal{P} \in \mathcal{N}} \text{sgn}_q(\mathcal{P}) w(\mathcal{P}) = 0.$$

To show this, we find a fixed-point free involution  $\pi : \mathcal{N} \rightarrow \mathcal{N}$  with the property that for every  $\mathcal{P} \in \mathcal{N}$ ,

$$\text{sgn}_q(\mathcal{P}) w(\mathcal{P}) = -\text{sgn}_q(\pi(\mathcal{P})) w(\pi(\mathcal{P})). \quad (1)$$

So suppose  $\mathcal{P} = \{P_1, \dots, P_k\} \in \mathcal{N}$ . Define  $\pi$  as follows. Let  $i$  be the minimum index of  $I$  such that  $P_i$  and  $P_{i+1}$  intersect (which exists by Lemma 4.2). Let  $x$  be the *last* vertex which they have in common. Let  $K_1 : r_i \Rightarrow x$  and  $L_1 : x \Rightarrow c_{\sigma_{\mathcal{P}}(i)}$  be the two subpaths of  $P_i$  such that  $P_i = K_1 L_1$ . Define  $K_2$  and  $L_2$  from  $P_{i+1}$  similarly.

Now we set

$$\pi(P_l) = \begin{cases} K_1 L_2 & \text{if } l = i, \\ K_2 L_1 & \text{if } l = i + 1, \\ P_l & \text{otherwise.} \end{cases}$$

It is clear from the definition that  $\pi$  is an involution without fixed points so it remains to prove Equation 1. Since  $\pi$  is an involution, we may assume

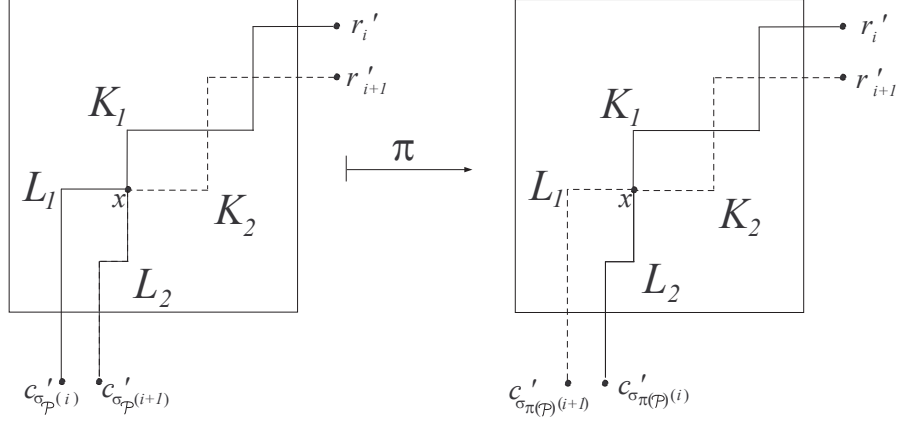


Figure 7: Example of how  $\pi$  acts on two intersecting paths. On the left hand side,  $P_i$  is the solid path and  $P_{i+1}$  is the dotted path. On the right hand side,  $\pi(P_i)$  is the solid path,  $\pi(P_{i+1})$  is the dotted path.

without loss of generality that  $\sigma_P(i) < \sigma_P(i+1)$ . Thus  $\sigma_{\pi(P)} = (i \ i+1)\sigma_P$  and so  $\sigma_{\pi(P)}$  has exactly one more inversion, i.e.,

$$l(\sigma_{\pi(P)}) = l(\sigma_P) + 1. \quad (2)$$

Now consider  $w(P_i)w(P_{i+1})$ . There are two cases to consider. First suppose  $L_2$  has a horizontal edge. Then we find that

$$\begin{aligned} w(P_i)w(P_{i+1}) &= w(K_1)w(L_1)w(K_2)w(L_2) \\ &= w(K_1)q w(K_2)w(L_1)w(L_2) \text{ (Lemma 3.5)} \\ &= w(K_1)q q w(K_2)w(L_2)w(L_1) \text{ (Lemma 3.6)} \\ &= w(K_1)q q q^{-1} w(L_2)w(K_2)w(L_1) \text{ (Lemma 3.5)} \\ &= q w(\pi(P_i))w(\pi(P_{i+1})). \end{aligned} \quad (3)$$

If  $L_2$  has only vertical edges then a similar calculation shows again that

$w(P_i)w(P_{i+1}) = q w(\pi(P_i))w(\pi(P_{i+1}))$ . Therefore

$$\begin{aligned} w(\mathcal{P}) &= \left( \prod_{j=1}^{i-1} w(P_j) \right) w(P_i)w(P_{i+1}) \left( \prod_{j=i+2}^k w(P_j) \right) \\ &= \left( \prod_{j=1}^{i-1} w(\pi(P_j)) \right) q w(\pi(P_i))w(\pi(P_{i+1})) \left( \prod_{j=i+2}^k w(\pi(P_j)) \right) \\ &= q w(\pi(\mathcal{P})). \end{aligned}$$

Finally, by Equations 2 and 3, we obtain

$$\begin{aligned} \operatorname{sgn}_q(\mathcal{P})w(\mathcal{P}) + \operatorname{sgn}_q(\pi(\mathcal{P}))w(\pi(\mathcal{P})) &= \\ (-q)^{l(\sigma_{\mathcal{P}})} q w(\pi(\mathcal{P})) + (-q)^{l(\sigma_{\mathcal{P}})+1} w(\pi(\mathcal{P})) &= 0. \end{aligned}$$

This proves Equation 1 and shows that

$$\det_q(M_{\mathcal{C}}[I, J]) = \sum_{\mathcal{P}} \operatorname{sgn}_q(\mathcal{P})w(\mathcal{P}),$$

where the sum is over all vertex disjoint  $(R_I, C_J)$  path systems.

By Proposition 3.3, we know that  $\mathcal{G}_{\mathcal{C}}$  is planar and so  $\mathcal{P}$  cannot have any edge crossings. This implies that we necessarily must have  $\mathcal{P} = \{P_l : l \Rightarrow l \mid l = 1, \dots, k\}$ . Thus  $\sigma_{\mathcal{P}}$  is the identity permutation and so  $\operatorname{sgn}_q(\mathcal{P}) = 1$ .

We therefore obtain the desired equation in the statement of the theorem, namely,

$$\det_q(M_{\mathcal{C}}[I, J]) = \sum_{\mathcal{P}} w(\mathcal{P}),$$

where the sum is over all vertex disjoint  $(R_I, C_J)$  path systems.  $\square$

**Theorem 4.5.** *Let  $\mathcal{C}$  be a Cauchon diagram and  $I \subseteq [m]$  and  $J \subseteq [n]$  be two subsets of the same size. Then  $\det_q(M_{\mathcal{C}}[I, J]) = 0$  if and only if there does not exist a vertex disjoint  $(R_I, C_J)$ -path system.*

*Proof.* If there does not exist a vertex disjoint  $(R_I, C_J)$ -path system then by Theorem 4.4,  $\det_q(M_{\mathcal{C}}[I, J])$  is the empty sum and so  $\det_q(M_{\mathcal{C}}[I, J]) = 0$

Conversely, suppose there exists at least one vertex disjoint  $(R_I, C_J)$ -path system. Let  $\mathcal{P}$  be one. Then  $w(\mathcal{P})$  consists of a nonempty sum of elements of the algebra  $\mathcal{B}$  where each summand is a product of  $\mathcal{B}$ -generators and their

inverses. By arranging each such product so the generators appear from left to right in lexicographic order, it follows that we can uniquely write

$$\begin{aligned}\det_q(M_{\mathcal{C}}[I, J]) &= \sum_{\mathcal{P}} w(\mathcal{P}) \\ &= \sum_{Q \subseteq [m] \times [n]} P_Q(q) \prod_{\alpha \in Q} t_{\alpha}^{r(\alpha, Q)},\end{aligned}\tag{4}$$

where  $P_Q(q)$  is some polynomial in  $\mathbb{Z}_{\geq 0}[q, q^{-1}]$ , and  $r(\alpha, Q)$  is an integer.

Since at least one vertex disjoint  $(R_I, C_J)$ -path system exists, the sum in Equation 4 is non-empty and so there exists at least one  $Q \subseteq [m] \times [n]$  such that  $P_Q \not\equiv 0$ . But since  $q$  is transcendental over  $\mathbb{Q}$ , we know that  $P_Q(q) \neq 0$  for any  $P_Q \not\equiv 0$ . Thus  $\det_q(M_{\mathcal{C}}[I, J]) \neq 0$ . □

## 5 Finding Vanishing Quantum Minors

Launois [12] proved the following.

**Theorem 5.1.** *Let  $q$  be transcendental over  $\mathbb{C}$ . Then the  $\mathcal{H}$ -invariant prime ideals of  $\mathcal{A}$  are generated by quantum minors of the matrix of canonical generators  $X_{\mathcal{A}}$ .*

Launois [11] has given an algorithm which takes as input the Cauchon diagram corresponding to an  $\mathcal{H}$ -invariant prime ideal  $I$ , and outputs a matrix whose vanishing quantum minors correspond to quantum minors of  $X_{\mathcal{A}}$  which generate  $I$ . This algorithm is essentially Cauchon's Deleting-Derivations Algorithm run in reverse.

**Algorithm 5.2.** (Note that the entries of every matrix below are from the algebra  $\mathcal{B}$  from Definition 2.4).

**Input** A Cauchon diagram  $\mathcal{C}$ .

**Output** A matrix  $T^{(m,n)}$  with entries from the algebra  $\mathcal{B}$ .

**Initialization** Let  $T^{(1,1)}$  be an  $m \times n$  matrix defined by

$$T^{(1,1)}[i, j] = \begin{cases} t_{i,j} & \text{if } (i, j) \in W_{\mathcal{C}}, \\ 0 & \text{if } (i, j) \in B_{\mathcal{C}}. \end{cases}$$

Set  $(s, t) = (1, 2)$  and let  $T^{(s,t)-}[i, j] := t_{i,j}^{(s,t)-}$ .

**While**  $(s, t) \neq (m, n + 1)$ , do the following:

1. Construct the matrix  $T^{(s,t)}$ , where  $T^{(s,t)}[i, j] := t_{i,j}^{(s,t)}$ , by

$$t_{i,j}^{(s,t)} = \begin{cases} t_{i,j}^{(s,t)-} + t_{i,s}^{(s,t)-} (t_{s,t}^{(s,t)-})^{-1} t_{r,j}^{(s,t)-} & \text{if } (s, t) > (i, j) \text{ and } t_{s,t} \neq 0, \\ t_{i,j}^{(s,t)-} & \text{otherwise.} \end{cases}$$

2. Set  $(s, t) = (s, t)^+$ .

**End while.**

Notice that we have  $t_{s,k}^{(s,t)} = t_{s,k}^{(1,1)}$  for all  $k \in [n]$ . Launois [12] proved the following.

**Theorem 5.3.** *Let  $\mathcal{I}$  be an  $\mathcal{H}$ -invariant prime ideal of  $\mathcal{A} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{C}))$ . Let  $\mathcal{C}$  be the Cauchon diagram associated to  $\mathcal{I}$ . Apply Algorithm 5.2 to  $\mathcal{C}$  to obtain the matrix  $T^{(m,n)}$ . If a square submatrix in  $T^{(m,n)}$  has a vanishing quantum minor, then the corresponding quantum minor in the matrix  $X_{\mathcal{A}}$  is a generator for  $\mathcal{I}$ . Furthermore,  $\mathcal{I}$  is generated by all such quantum minors.*

On the other hand, we prove the following.

**Lemma 5.4.** *Let  $\mathcal{C}$  be a Cauchon diagram. Then the path matrix  $M_{\mathcal{C}}$  is the same as the matrix obtained at the end of Algorithm 5.2.*

Before proving this lemma in its full generality, we apply Algorithm 5.2 to a small Cauchon diagram, and compare the result with the corresponding path matrix.

**Example 5.5.** Consider the  $3 \times 3$  Cauchon diagram  $\mathcal{C}$  in Figure 8. The initialization step of Launois's algorithm gives

$$T^{(1,1)} = \begin{bmatrix} t_{1,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & t_{3,2} & t_{3,3} \end{bmatrix}.$$

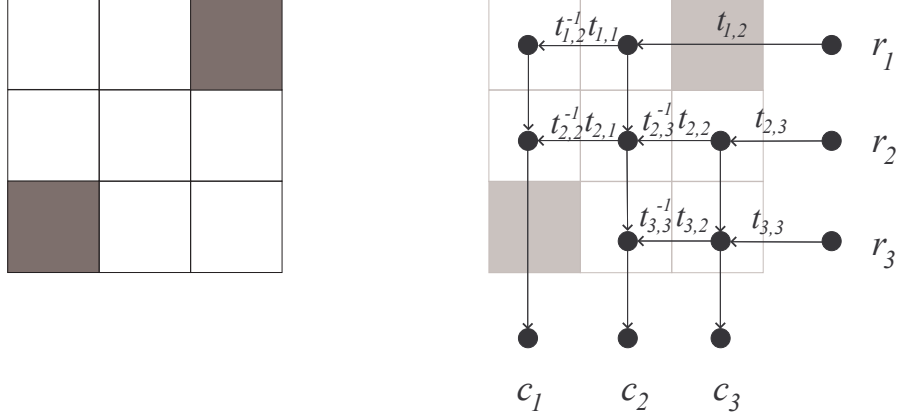


Figure 8: Example 5.5: A Cauchon diagram on the left, and its Cauchon graph on the right (the unit weights on the vertical edges are not shown)

Notice that each non-zero entry  $t_{i,j}$  is precisely the weight of the path  $r_i \rightarrow (i,j) \rightarrow c_j$  in  $\mathcal{G}_C$ .

Now at step  $(s,t)$  of the algorithm, the only entries of  $T^{(s,t)}$  that are modified from the previous step are those which are “north-west” of  $(s,t)$ . In particular, steps  $(s,t)$  with either  $s = 1$  or  $t = 1$  do not change the previous matrix. In our example then, we have  $T^{(1,1)} = T^{(1,2)} = T^{(1,3)} = T^{(2,1)}$ .

At step  $(2,2)$ , the only entry north-west of this entry is  $(1,1)$ . We therefore obtain

$$T^{(2,2)} = \begin{bmatrix} t_{1,1} + t_{1,2}t_{2,2}^{-1}t_{2,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & t_{3,2} & t_{3,3} \end{bmatrix}.$$

Notice that the new value of entry  $(1,1)$  is precisely the weight of the path  $r_1 \rightarrow (1,1) \rightarrow c_1$  plus the weight of the path  $r_1 \rightarrow (1,2) \rightarrow (2,2) \rightarrow (2,1) \rightarrow c_1$ .

The next step is  $(2,3)$  and entry  $(2,3)$  of  $T^{(2,2)}$  is non-zero. However, the entries north-west of  $(2,3)$  are  $(1,1)$  and  $(1,2)$ , and since  $T^{(2,2)}[1,3] = t_{1,3}^{(2,2)} = 0$ , the net effect of the algorithm at this step is to change nothing. Thus  $T^{(2,3)} = T^{(2,2)}$ .

We also have  $T^{(3,1)} = T^{(2,3)}$ , so consider step  $(3,2)$ . By similar reasoning as in step  $(2,3)$ , we find  $T^{(3,1)} = T^{(3,2)}$ . The last step is  $(3,3)$ . Applying the

algorithm we get

$$T^{(3,3)} = \begin{bmatrix} t_{1,2}t_{2,2}^{-1}t_{2,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} + t_{2,3}t_{3,3}^{-1}t_{3,2} & t_{2,3} \\ 0 & t_{3,2} & t_{3,3} \end{bmatrix}.$$

As one can easily verify,  $T^{(3,3)}$  is precisely the path matrix  $M_{\mathcal{C}}$ .

*Proof of Lemma 5.4.* Fix  $n$ . We prove the lemma by induction on the number of rows  $m$ . As in Algorithm 5.2, we will denote  $T^{(s,t)}[i, j]$  by  $t_{i,j}^{(s,t)} \in \mathcal{B}$

First note that since we only modify entries which are north-west of the entry corresponding to the current step, the algorithm will always leave the  $m$ th row unmodified. That is,  $T^{(s,t)}[m, [n]] = T^{(1,1)}[m, [n]]$  for all  $(s, t)$ . But by the algorithm we get that for  $k \in [n]$ ,

$$t_{m,k}^{(m,n)} = t_{m,k}^{(1,1)} = \begin{cases} t_{m,k} & \text{if } (m, k) \in W_{\mathcal{C}}, \\ 0 & \text{if } (m, k) \in B_{\mathcal{C}}. \end{cases}$$

Now in the  $m$ th row of  $\mathcal{G}_{\mathcal{C}}$ , there is clearly at most one possible path from  $r_m$  to  $c_k$ . This path exists if and only if  $(m, k)$  is a white square, and by Lemma 3.3, this path has weight exactly  $t_{m,k}$ .

From these two observations we see that the  $m$ th row in  $T^{(m,n)}$  is exactly the same as the  $m$ th row in  $M_{\mathcal{C}}$ . Similarly, the  $n$ th column of  $T^{(m,n)}$  is equal to the  $n$ th column of  $M_{\mathcal{C}}$ . In particular the lemma is true when  $m = 1$ .

So suppose the lemma is true for all Cauchon diagrams with less than  $m$  rows. If we obtain the Cauchon diagram  $\mathcal{C}'$  from  $\mathcal{C}$  by deleting the  $m$ th row, then by induction we have  $T^{(m-1,n)} = M_{\mathcal{C}'}$ . An equivalent way of stating this is that if  $i < m$ ,  $t_{i,j}^{(m-1,n)}$  is the total of the weights of all paths in  $\mathcal{G}_{\mathcal{C}}$  from  $r_i$  to  $c_j$  which *do not use* a horizontal edge in row  $m$ .

As we already noted,  $T^{(m,n)}[m, [n]] = M_{\mathcal{C}}[m, [n]]$  and  $T^{(m,n)}[[m], n] = M_{\mathcal{C}}[[m], n]$  so to complete the proof, we establish the following claim by induction on  $k \in [n]$  where  $k$  will denote the  $k$ th column of  $\mathcal{C}$ . It will follow from this that  $T^{(m,n)} = M_{\mathcal{C}}$ . (At this point, the reader may wish to review the while loop in Algorithm 5.2).

**Claim.** If  $(i, j) < (m, k)$ , then  $t_{i,j}^{(m,k)}$  is obtained from  $t_{i,j}^{(m,k)-}$  by adding the weights of all paths  $P$  that satisfy the following properties:

1.  $P$  is a path from  $r_i$  to  $c_j$ .

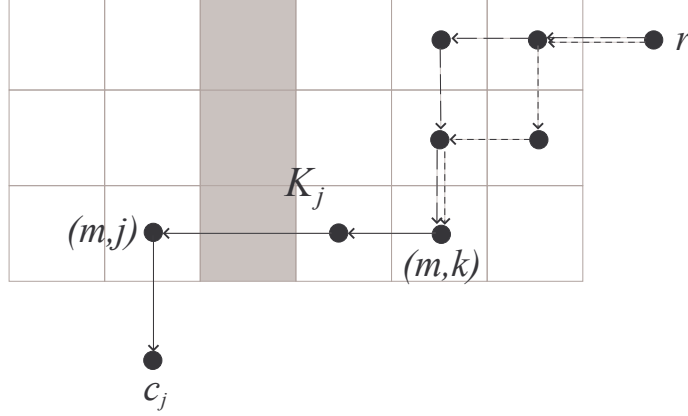


Figure 9:  $K_j$  is the path consisting of the edges drawn with a solid line. Concatenating  $K_j$  with either of the paths drawn with dotted edges, gives a path which satisfies the properties in the claim.

2.  $P$  contains the subpath  $K_j : (m, k) \Rightarrow c_j$ . Note that  $K_j$  consists of horizontal edges from  $(m, k)$  to  $(m, j)$  and then the vertical edge  $((m, j), c_j)$ .
3.  $P$  contains a vertical edge  $((l, k), (m, k))$  for some  $l < m$ . In other words, if  $k' > k$ , then vertex  $(m, k')$  (if it exists) is *not* an internal vertex of  $P$ .

For  $k = 1$ , we know that  $T^{(m,1)} = T^{(m-1,n)}$ . On the other hand, since there are no  $j < k$  the claim is trivially true.

So suppose  $k > 1$ , and the claim is true for step  $(m, k - 1)$ .

If  $(m, k)$  is black in  $\mathcal{C}$ , then according to Algorithm 5.2, we set  $t_{i,j}^{(m,k)} = t_{i,j}^{(m,k-1)}$ . On the other hand, if  $(m, k)$  is black then  $K_j$  can not exist for any  $j$  and so there are no paths which satisfy the properties in the claim. This then proves the claim in the case that  $(m, k)$  is black.

So suppose  $(m, k)$  is white. If  $(m, j)$  is black for some  $j < k$ , then  $t_{m,j}^{(m,k-1)} = t_{m,j}^{(1,1)} = 0$  and so by the algorithm, we again have  $t_{i,j}^{(m,k)} = t_{i,j}^{(m,k-1)}$ . On the other hand, if  $(m, j)$  is black, then  $K_j$  can not exist. This proves the statement in the claim for those  $j < k$  such that  $(m, j)$  is black.

Finally, suppose  $(m, k)$  and  $(m, j)$  are white squares. Then  $K_j$  exists in  $\mathcal{G}_{\mathcal{C}}$ . Note that  $w(K_j) = t_{m,k}^{-1} t_{m,j}$ . On the other hand, if  $P$  is a path satisfying



the properties in the claim, then there is a path  $P' : r_i \Rightarrow (m, k)$  such that  $P = P'K_j$ . But by Property 3, the last edge in  $P'$  is vertical. So if we concatenate  $P'$  with the vertical path  $L_k : (m, k) \Rightarrow c_k$ , we get a path (with the same weight as  $P'$ ) from  $r_i$  to  $c_k$  which does not use any horizontal edge in the last row.

By induction, the set of all such  $P'$  has total weight  $t_{m,k}^{(m-1,k)}$ . But this entry has not been modified at step  $(m, l)$  of the algorithm for any  $l < k$ , so in fact, the set of all such  $P'$  has total weight  $t_{m,k}^{(m,k-1)}$ . Hence the total weight of all  $P$  that satisfy the properties in the claim is exactly

$$t_{m,k}^{(m,k-1)} w(K_j) = t_{m,k}^{(m,k-1)} t_{m,k}^{-1} t_{m,j}.$$

On the other hand, by Algorithm 5.2,

$$t_{i,j}^{(m,k)} = t_{i,j}^{(m,k-1)} + t_{m,k}^{(m,k-1)} t_{m,k}^{-1} t_{m,j}.$$

This finishes the proof of the claim and the lemma. □

Now we state the main result of this paper, which follows immediately from Theorem 4.5, Theorem 5.3 and Lemma 5.4.

**Theorem 5.6.** *Let  $\mathcal{C}$  be an  $m \times n$  Cauchon diagram corresponding to the  $\mathcal{H}$ -invariant prime ideal  $\mathcal{I}$ . Then a quantum minor  $\det_q(X_{\mathcal{A}}[I, J])$  of  $X_{\mathcal{A}}$  is in  $\mathcal{I}$  if and only if there does not exist a vertex disjoint  $(R_I, C_J)$ -path system in the Cauchon graph  $\mathcal{G}_{\mathcal{C}}$ .*

We should emphasize that in general, neither the preceding theorem nor Theorem 5.3 give a *minimal* generating set; further work must be done to find one.

## 6 Concluding Remarks

We note that Algorithm 5.2 can, in general, result in a matrix which has entries with exponentially many terms. This algorithm is therefore not always ideal if one simply wishes to check whether a specific quantum minor appears in the generating set given in Theorem 5.1. On the other hand, finding a vertex-disjoint path system in a Cauchon graph is computationally efficient as it is a special case of Menger's Theorem in graph theory, which is well-known to be implementable in polynomial time.

The question of determining which  $\mathcal{H}$ -invariant prime ideals are primitive has received recent attention. Bell, Launois and Nguyen [3] have completely determined which  $2 \times n$  Cauchon diagrams correspond to primitive  $\mathcal{H}$ -invariant prime ideals. For  $m > 2$  however, the question seems to be much harder although progress has been made by Bell, Launois and Lutley [2] on a method which, in principal, can solve the  $m \times n$  case. In particular, they explicitly give a method which can determine primitivity of  $\mathcal{H}$ -primes in  $\mathcal{O}_q(\mathcal{M}_{3,n}(\mathbb{C}))$  using finite-state automata. It would be interesting if an approach to this problem can be developed either via the results of Goodearl, Launois and Lenagan [7] or the vertex-disjoint paths method of this paper.

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